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Hard squares with diagonal attractions

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Abstract. We complete the solution of the square lattice gas with nearest neighbour exclusion and attractive next-nearest neighbour (diagonal) interactions on the special surface corresponding to regimes III and IV of the generalised hard hexagon model. The interfacial tension, correlation length and sublattice density difference are calculated throughout these regimes by obtaining the eigenvalues of the row-to-row and corner transfer matrices. The associated critical exponents are found to be $\mu = \nu = \frac{5}{4}$, $\beta = \frac{3}{32}$ in regime III and $\mu' = \nu' = \frac{5}{2}$, $\beta' = \frac{1}{4}$ in regime IV. In particular, our results confirm the recent proposal by Huse that regime III is the first-order coexistence surface (separating the disordered fluid phase from the square ordered solid phases) and that the regime III/regime IV boundary is a line of tricritical points.

1. Introduction

In previous papers (Baxter 1980, 1981b, Baxter and Pearce 1982; see also Baxter 1982) attention has been focused on various aspects of the exact solution to the generalised hard hexagon model. This model is actually a hard square lattice gas with diagonal interactions L and M related to the activity z by the constraint

$$z = (1-e^{-L})(1-e^{-M})/(e^{L+M}-e^{L}-e^{M}). \tag{1.1}$$

The hard hexagon model is obtained by taking the limit $L = 0$, $M = -\infty$ or *vice versa*. Here, however, we wish to focus our attention on hard squares with *attractive* diagonal interactions $L,M > 0$. In this case, the special soluble surface (1.1) corresponds to regimes III and IV of the generalised hard hexagon model (Baxter 1980).

Up till now interest in the generalised hard hexagon model has centred on regimes I and II that pertain to hard hexagons. Very recently, however, Huse (1982) has proposed that, for interacting hard squares, regime III actually coincides with the *first-order* coexistence surface separating the disordered fluid phase from the square ordered solid phases. Moreover, Huse argues that the critical boundary dividing regimes III and IV must be a line of *tricritical* points and, for the typical case when $L = M$, puts forward the phase diagram shown in figure 1. This fortuitous and unexpected situation (previously regime III has only been approached from the disordered phase) renders regimes III and IV extremely interesting. In this paper, we confirm in detail the picture put forward by Huse. In particular, by obtaining the eigenvalues of the row-to-row and corner transfer matrices, we are able to calculate the interfacial tension, the correlation length and the sublattice densities throughout regimes III and IV.

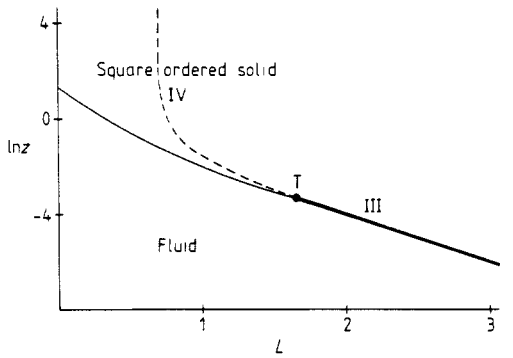

Figure 1. The phase diagram for the hard square lattice gas with activity z and diagonal attractions $L = M$. The soluble regime III (heavy full curve) is a line of triple points terminating at a tricritical point T. Along this curve the fluid phase coexists with the two square ordered solid phases. The soluble regime IV (broken curve) is the analytic continuation of regime III but lies entirely in the square ordered solid phase. For L less than its tricritical value there is a continuous transition between the fluid and solid phases. The locus of this transition (shown schematically by the light full curve) intersects the ln z axis at $z \approx 3.796$ corresponding to non-interacting hard squares (Gaunt and Fisher 1965, Baxter *et al* 1980) and meets regime III at the tricritical point T.

We shall use the notation of previous work (Baxter 1980, 1981b, 1982, Baxter and Pearce 1982): x and w are elliptic function parameters that are defined differently in regimes III and IV; Δ, t are variables defined for both regimes by

$$\begin{aligned}\Delta &= z^{1/2}(1 - z\,\mathrm{e}^{L+M}) \\ &= -\Delta_c \prod_{n=1}^{\infty} \left(\frac{1-2t^n\cos(2\pi/5)+t^{2n}}{1-2t^n\cos(4\pi/5)+t^{2n}}\right)^{5/2} \\ &= -\Delta_c[1 - \tfrac{5}{2}\sqrt{5}t + \tfrac{5}{8}(25-2\sqrt{5})t^2 + \ldots] \end{aligned} \tag{1.2}$$

where $\Delta_c = [\frac{1}{2}(\sqrt{5}-1)]^{5/2} = 0.30028\ldots$ and $-1 < t < 1$. This t is the variable q^2 of Baxter (1980): it vanishes linearly at the tricritical point $\Delta = -\Delta_c$, being there an analytic function of Δ. Thus t can be taken as the 'deviation from criticality' variable: it is positive in regime III and negative in regime IV.

The layout of the remainder of the paper is as follows. In § 2, we solve the functional equations for the eigenvalues of the row-to-row transfer matrix in regime III and obtain the interfacial tension and correlation length. In § 3, we do the same for regime IV. Finally, in § 4, we present the results for the sublattice densities in these regimes.

2. Transfer matrix eigenvalues: regime III

2.1. The eigenvalue spectrum

In Baxter and Pearce (1982) we set up functional equations for the eigenvalues of the transfer matrix $V(w)$. In regimes II and III these equations take the form

$$T(w)T(xw) = 1 + T(x^3w), \qquad T(x^5w) = T(w), \tag{2.1a, b}$$

where

$$T(w)=[-f(x)f(x^2w)/f(xw)f(x^3w)]^N V(w) \tag{2.1c}$$

and $f(w)\equiv\acute{f}(w,x^5)$ with the elliptic function $f(w,q)$ of nome q being defined by

$$f(w,q)=\prod_{n=1}^{\infty}(1-q^{n-1}w)(1-q^n w^{-1})(1-q^n) \tag{2.2}$$

for any q in the range $-1<q<1$. The parameters x and w are related to the interactions L and M by the equations

$$e^L=\frac{wf(x)f(x^2w)f(x^3w)}{xf(x^2)f^2(w)}, \qquad e^M=\frac{f(x)f(xw)f(x^2w)}{wf(x^2)f^2(x^4w)}. \tag{2.3a,b}$$

In the physical regimes x and w lie in the ranges

$$\text{Regime II} \qquad 0<x<1, \qquad 1<w<x^{-1}, \tag{2.4a}$$

$$\text{Regime III} \qquad 0<x<1, \qquad x<w<1. \tag{2.4b}$$

In general, however, we regard x as a given constant and w as a complex variable.

The eigenvalue equations (2.1) do not distinguish between regimes II and III; their solutions will be eigenvalues in both regimes. Following our previous analysis of regime II (Baxter and Pearce 1982), we conclude that the regime III eigenvalues must be of the form

$$V_{p,s}(w)=Rw^n\prod_{i=1}^{p}f(xw/a_i)f(x^3w/a_i')$$

$$\times\prod_{j=1}^{r}f(x^{1/2}w/b_j)f(x^{7/2}w/b_j')\prod_{k=1}^{s}f(x^2w/c_k)\prod_{l=1}^{t}f(x^{9/2}w/d_l) \tag{2.5a}$$

where R is complex, a_i, a_i', b_j, b_j', c_k, d_l are complex numbers locating the zeros of $V_{p,s}(w)$ and n, p, r, s, t are non-negative integers satisfying

$$2p+2r+s+t=N, \qquad t=2n. \tag{2.5b}$$

In practice, the regime III eigenvalues are the most difficult to analyse. Unlike other regimes, the complex numbers a_i, a_i', b_j, b_j', etc, need not all be unimodular in the limit $N\to\infty$. In addition, the zeros need not occur in pairs so that, in general, $a_i\neq a_i'$ and $b_j\neq b_j'$.

As in regime II, there is in fact another constraint on the integers n, p, r, s and t. If we assume that $n>0$ and substitute (2.5) into the RHS of (2.1a) with $w=ax^{-3}$, we find that it vanishes for n choices of a. The LHS, however, vanishes if and only if $a=a_i$ with $i=1,2,\ldots,p$ or $a=c_k$ with $k=1,2,\ldots,s$. Hence either

$$n=p+s \qquad \text{or} \qquad n=0. \tag{2.6}$$

In regime II, $n=O(N/3)$ as $N\to\infty$ for the largest eigenvalues so the exceptional case $n=0$, $p+s\neq 0$ did not arise.

From perturbation expansions we expect that for the *largest* eigenvalues in regime III

$$V(w)=O(1) \qquad \text{as } N\to\infty, \qquad x<|w|<1. \tag{2.7}$$

For the moment, let us exclude the exceptional case $n=0$, $p+s\neq 0$ and concentrate

on the 'regular' eigenvalues for which $n = p + s$. For these eigenvalues, it follows from (2.7) that $n = p + s = O(1)$ as $N \to \infty$ and hence for the largest regular eigenvalues in regime III

$$n = p + s, \qquad r = \tfrac{1}{2}(N - 4p - 3s), \qquad s = N \pmod 2, \qquad t = 2(p + s). \tag{2.8}$$

In general, for given p and s, there are many eigenvalues of the form given by (2.5) and (2.8). The integers p and s therefore label *bands* of eigenvalues.

To solve the transfer matrix equation (2.1) we begin by observing, from (2.5), that for large N

$$\begin{aligned} &|T(w)| = O(1), && x^{3/2} < |w| < x^{-1/2}, \\ &|T(w)| = O(x^{-\varepsilon N}) \gg 1, && x^{5/2} < |w| < x^{3/2},\ x^{9/2} < |w| < x^{7/2}, \\ &|T(w)| = O(x^{\varepsilon N}) \ll 1, && x^{7/2} < |w| < x^{5/2}, \end{aligned} \tag{2.9}$$

with $\varepsilon > 0$. Next we define $K(w)$ by

$$T_{p,s}(w) = w^{p+s} K(w) \prod_{l=1}^{2(p+s)} (1 - x^{1/2} d_l w^{-1}), \qquad x^{3/2} < |w| < x^{-1/2}. \tag{2.10}$$

From (2.5), $\ln K(w)$ is then analytic and Laurent expansible in the annulus $x^{3/2} < |w| < x^{-1/2}$. We now consider the smaller annulus $x^{1/2} < |w| < x^{-1/2}$ and substitute (2.10) into (2.1a) ignoring the exponentially small term on the RHS. Taking logarithms, Laurent expanding and equating coefficients, we can solve for $\ln K(w)$. In this way we find

$$T_{p,s}(w) = \prod_{l=1}^{2(p+s)} \psi(w/d_l), \qquad x^{3/2} < |w| < x^{-1/2}, \tag{2.11a}$$

where

$$\psi(w) = w^{1/2} f(x^{1/2} w^{-1}, x^2) / f(x^{1/2} w, x^2). \tag{2.11b}$$

To complete the solution for the regular eigenvalues in regime III we need equations for the complex numbers d_l appearing in (2.11). Setting $w = dx^{-1/2}$ in (2.1a), we find that

$$1 + T_{p,s}(x^{5/2} d) = 0 \tag{2.12}$$

for $d = d_1, d_2, \ldots, d_l$ or $d = b_1, b_2, \ldots, b_r$. These equations, however, involve the solution for $T_{p,s}(w)$ in the annulus $x^{9/2} < |w| < x^{3/2}$, which can only be obtained by solving (2.1a) in the annulus $x^{7/2} < |w| < x^{3/2}$. Unfortunately, from (2.9), neither term on the RHS of (2.1a) can be ignored in this annulus so we must solve the equation

$$T(w) T(xw) = 1 + \prod_{l=1}^{2(p+s)} \psi(w/d_l), \qquad x^{7/2} < |w| < x^{3/2}. \tag{2.13}$$

In general we have been unable to solve this equation.

2.2. Interfacial tension

If N is even, the regular eigenvalues of largest modulus are obtained by setting $(p, s) = (0, 0) = \mathbf{0}$ in (2.11). This gives

$$T_{\mathbf{0}}(w) = 1, \qquad x^{3/2} < |w| < x^{-1/2}, \tag{2.14}$$

which agrees with the partition function per site previously obtained (Baxter 1980) by the matrix inversion trick. To solve for $T_0(w)$ in the annulus $x^{9/2}<|w|<x^{3/2}$ (i.e. outside the physical regime) we define $L(w)$ by

$$T_0(w)=w^{-N/2}L(w)(1-x^3w^{-1})^N/(1-x^2w^{-1})^N(1-x^4w^{-1})^N. \tag{2.15}$$

From (2.5), $\ln L(w)$ is analytic and Laurent expansible in the annulus $x^{9/2}<|w|<x^{3/2}$. Now consider the smaller annulus $x^{7/2}<|w|<x^{3/2}$. Using (2.14) we see that (2.1a) becomes

$$T_0(w)T_0(xw)=2, \qquad x^{7/2}<|w|<x^{3/2}. \tag{2.16}$$

Substituting (2.15) into (2.16), taking logarithms, Laurent expanding and equating coefficients as before, we find that

$$T_0(w)=\pm\sqrt{2}\bar{\psi}^N(w), \qquad x^{9/2}<|w|<x^{3/2}, \tag{2.17}$$

where

$$\bar{\psi}(w)=\mathrm{i}\psi(w/x^{1/2})=(-w/x^{1/2})^{1/2}f(xw,x^2)/f(w,x^2). \tag{2.18}$$

Thus far, for N even, we have found two largest eigenvalues which are given in a period annulus by

$$T_{0;\pm}=\begin{cases}1, & x^{3/2}<|w|<x^{-1/2},\\ \pm\sqrt{2}\bar{\psi}^N(w), & x^{9/2}<|w|<x^{3/2}.\end{cases} \tag{2.19}$$

We now have to go back and look at the exceptional case $n=0$, $p+s\neq 0$. In this case, we find a *single* 'exceptional' eigenvalue given by

$$V_X(w)=\begin{cases}R\prod_{i=1}^{N/2}f(xw/a_i)f(x^3w/a_i'), & N \text{ even},\\ Rf(x^2w/c)\prod_{i=1}^{(N-1)/2}f(xw/a_i)f(x^3w/a_i'), & N \text{ odd}.\end{cases} \tag{2.20}$$

For this eigenvalue, when N is large

$$\begin{aligned}&|T_X(w)|=\mathrm{O}(1), && x^2<|w|<x^{-1},\\ &|T_X(w)|=\mathrm{O}(x^{\epsilon N})\ll 1, && x^4<|w|<x^2.\end{aligned} \tag{2.21}$$

Hence the LHS of (2.1a) is exponentially smaller than one for w in the annulus $x^4<|w|<x$. It follows that

$$T_X(w)=-1, \qquad x^2<|w|<x^{-1}. \tag{2.22}$$

Although it is difficult to obtain $T_X(w)$ in the annulus $x^4<|w|<x^2$, it can easily be shown that there is only *one* eigenvalue of the form (2.20) using small x expansions.

In total, for N even, we have thus found *three* eigenvalues of largest modulus in regime III. In the large N limit, these are given by (2.19) and (2.22). The associated eigenvectors can be identified with linear combinations of *three* pure phases: fluid plus the two square ordered phases. Thus all *three* of these phases coexist in regime III! By the Perron–Frobenius theorem, the largest eigenvalue of the transfer matrix cannot be degenerate in the physical regime $x<w<1$ for *finite* N. In this regime the three eigenvalues $T_{0;+}(w)$, $T_{0;-}(w)$, $T_X(w)$ are therefore asymptotically degenerate as

$N \to \infty$. In the limit of large N, we in fact expect (Fisher 1969) that

$$V_{\mathbf{0};-}(w)/V_{\mathbf{0};+}(w) = T_{\mathbf{0};-}(w)/T_{\mathbf{0};+}(w) = 1 - O(e^{-N\beta\sigma}) \tag{2.23}$$

where $\beta = 1/k_B T$ is the inverse temperature and σ is the interfacial tension.

To solve for the interfacial tension σ, we repeat the previous calculation of $T_{\mathbf{0};\pm}(w)$ in the annulus $x^{3/2} < |w| < x^{-1/2}$, keeping both terms on the RHS of (2.1a) and treating the smaller term as a correction. For $x < |w| < 1$ this gives

$$\ln T_{\mathbf{0};\pm}(w) = \frac{1}{2\pi i} \oint_{|w'|=1} \frac{dw'}{w'} J(w/w') \ln[1 \pm \sqrt{2}\, \bar{\psi}^N(xw')] \tag{2.24}$$

where

$$J(w) = w\bar{\psi}'(w)/\bar{\psi}(w) \tag{2.25}$$

and $\bar{\psi}'(w)$ is the derivative of $\bar{\psi}(w)$.

From (2.23) the interfacial tension is now given by

$$-\beta\sigma = \lim_{N\to\infty} N^{-1} \ln \ln[T_{\mathbf{0};-}(w)/T_{\mathbf{0};+}(w)]. \tag{2.26}$$

But for $x^{3/2} < |w| < x^{1/2}$, we have $|\bar{\psi}(w)| < 1$. Hence, for w' in the annulus $x^{1/2} < |w'| < x^{-1/2}$ and N large, we find

$$\ln[1 \pm \sqrt{2}\, \bar{\psi}^N(xw')] \sim \pm\sqrt{2}\, \bar{\psi}^N(xw'). \tag{2.27}$$

Also, since N is even, $\bar{\psi}^N(w)$ is analytic in the annulus $x^2 < |w| < 1$. For large N, we can therefore put (2.27) into (2.24) and integrate by steepest descents. Considered as a function of a real variable, $|\bar{\psi}(w)|$ has a minimum at $w = -x$. In the complex plane this corresponds to a saddle point of $|\bar{\psi}(w)|$. Noting that the contour $|w'| = 1$ in (2.24) passes through this saddle point at $w' = -1$, and using (2.26), we finally obtain

$$-\beta\sigma = \ln \bar{\psi}(-x) = \ln[x^{1/4} f(-1, x^2)/f(-x, x^2)] = \tfrac{1}{2} \ln k(x) \tag{2.28}$$

where $k(x)$ is the elliptic modulus given by

$$k(x) = 4x^{1/2} \prod_{n=1}^{\infty} \left(\frac{1 + x^{2n}}{1 + x^{2n-1}} \right)^4. \tag{2.29}$$

In the limit $x \to 1-$, i.e. at the tricritical point, the interfacial tension vanishes. The departure from criticality is conveniently measured by the conjugate nome t which is related to x in regime III by the equations

$$x = \exp(-4\pi^2/5\varepsilon), \qquad t = e^{-\varepsilon}. \tag{2.30}$$

The explicit relation between t and the interactions is given by (1.2). Transforming to the conjugate modulus (Baxter 1982, ch 15), (2.28) becomes

$$-\beta\sigma = \tfrac{1}{2} \ln k'(t^{5/4}) \tag{2.31}$$

with

$$k'(x) = \prod_{n=1}^{\infty} \left(\frac{1 - x^{2n-1}}{1 + x^{2n-1}} \right)^4. \tag{2.32}$$

In particular, near the tricritical point we find

$$\beta\sigma \sim 4t^{5/4}, \qquad t \to 0+. \tag{2.33}$$

Hence the interfacial tension exponent is

$$\mu = \tfrac{5}{4}. \tag{2.34}$$

The same result (2.28) for the interfacial tension can also be obtained by considering the asymptotic degeneracy between the exceptional eigenvalue $T_X(w)$ and one of the regular eigenvalues $T_{0;\pm}(w)$. Although the analysis is much more difficult, the same result is obtained because, for large N, the corrections to the exceptional eigenvalue $T_X(w)$ are much smaller than the corrections calculated for $T_{0;\pm}(w)$. Physically there is just one interfacial tension, between an ordered phase and the fluid; two ordered phase domains are always separated by fluid.

2.3. Correlation length

For N even, we have seen that there are three asymptotically degenerate maximum eigenvalues of the transfer matrix in regime III. For large N, the next-largest band of eigenvalues are given by setting $p = 1$, $s = 0$ in (2.11), i.e.

$$T_{1,0}(w) = \psi(w/d_1)\psi(w/d_2), \qquad x^{3/2} < |w| < x^{-1/2}. \tag{2.35}$$

In this case the complex numbers d_1, and d_2 will in fact be unimodular. Noting that $|\psi(w)| < 1$ for $x < |w| < 1$, we then see that in the physical regime there is a gap between the largest and next-largest eigenvalues.

The correlation length in the physical regime can now be obtained by integrating over the complete band of complex next-largest eigenvalues as in regimes I and II. To do this we first derive equations for d_1 and d_2 in (2.35). From (2.12), $d = d_1$ and $d = d_2$ must be two solutions of the equation

$$1 + T_{1,0}(x^{5/2}d) = 0. \tag{2.36}$$

But now to find $T_{1,0}(w)$ in the annulus $x^{9/2} < |w| < x^{3/2}$ we must solve the transfer matrix equation (2.1*a*) in the annulus $x^{7/2} < |w| < x^{3/2}$. Using (2.35) and the fact that $\psi(x^2 w) = \psi(w)$, this equation becomes

$$T_{1,0}(w)T_{1,0}(xw) = 1 + \psi(w/d_1)\psi(w/d_2). \tag{2.37}$$

To proceed we need to factorise the RHS of (2.37). This is achieved by using the identity

$$1 + \frac{\varepsilon w}{d_1^{1/2} d_2^{1/2}} \frac{f(x^{1/2}d_1/w, x^2) f(x^{1/2}d_2/w, x^2)}{f(x^{1/2}w/d_1, x^2) f(x^{1/2}w/d_2, x^2)}$$
$$= \frac{f(x, x^2) f(\varepsilon x^{1/2} d_2^{1/2}/d_1^{1/2}, x) f(-\varepsilon w/d_1^{1/2} d_2^{1/2}, x)}{f(x^{1/2}w/d_1, x^2) f(x^{1/2}w/d_2, x^2)} \tag{2.38}$$

which for $\varepsilon = \pm 1$ can be proved by standard techniques (Baxter 1982, ch 15). Next we define $L(w)$ in the annulus $x^4 < |w| < x^2$ by

$$T_{1,0}(w) = w^{N/2} L(w)(1 - x^3 w^{-1})^N \tag{2.39}$$

so that, from (2.5), $\ln L(w)$ is analytic therein. If we now consider the smaller annulus $x^3 < |w| < x^2$, substitute (2.38) and (2.39) into (2.37), take logarithms, Laurent expand and equate coefficients we can solve for $\ln L(w)$ as before. Doing this, we eventually

find that

$$T_{1,0}(w) = \pm\bar{\psi}^N(w)[f(x,x^2)f(\varepsilon x^{1/2}d_2^{1/2}/d_1^{1/2},x)]^{1/2}f(-\varepsilon w/x^2 d_1^{1/2}d_2^{1/2},x^2) \times M(w/x^{5/2}d_1)M(w/x^{5/2}d_2), \qquad x^{9/2}<|w|<x^{3/2}, \tag{2.40}$$

where

$$\ln M(w) = \sum_{n=1}^{\infty} \frac{x^n w^n + x^{2n} w^{-n}}{n(1-x^{2n})(1+x^n)}. \tag{2.41}$$

From (2.36) and (2.40) we conclude that d_1 and d_2 are two solutions of the equation

$$d^{N/2}\frac{f^N(x^{1/2}/d,x^2)}{f^N(x^{1/2}d,x^2)}[f(x,x^2)f(\varepsilon x^{1/2}d_2^{1/2}/d_1^{1/2},x)]^{1/2} \times f(-\varepsilon x^{1/2}d/d_1^{1/2}d_2^{1/2},x^2)M(d/d_1)M(d/d_2) = \pm 1. \tag{2.42}$$

Since the LHS of this equation is in fact unimodular when $|d_1| = |d_2| = 1$ and $d = d_1$ or $d = d_2$, this equation is consistent with our assumption that d_1 and d_2 are unimodular. Indeed, for small x, (2.42) admits $4\binom{N/2}{2}$ distinct solutions for d_1 and d_2 with $|d_1| = |d_2| = 1$. This is the number of ways of dividing a row of N sites, with N even and periodic boundary conditions, into two domains—one with no particles (vacuum) and one with particles on alternate sites (square ordered).

For N even, the solutions of (2.42) give a band of $4\binom{N/2}{2}$ complex next-largest eigenvalues of the form (2.35). We shall now assume, as seems reasonable from small x expansions, that in the limit $N\to\infty$ the solutions of (2.42) form a continuous distribution on the circles $|d_1| = 1$, $|d_2| = 1$ with a density $\rho(d_1, d_2)$. We now repeat the arguments outlined for calculating the correlations in regimes I and II (Baxter and Pearce 1982). In both the fluid and square ordered phases we find that the asymptotic behaviour of the correlation between sites separated by l rows, for l large, is

$$\langle\sigma_0\sigma_l\rangle - \langle\sigma_0\rangle\langle\sigma_l\rangle \sim \left(\frac{1}{4\pi \mathrm{i}}\right)^2 \oint_{|d_1|=1}\frac{\mathrm{d}d_1}{d_1}\oint_{|d_2|=1}\frac{\mathrm{d}d_2}{d_2}\rho(d_1,d_2)\psi^l(w/d_1)\psi^l(w/d_2). \tag{2.43}$$

Here the contours are closed loops formed by two circuits about the origin on the Riemann surface of two sheets on which the square root function is analytic. Except for poles on each sheet corresponding to $w = x^{-1/2}, x^{3/2}$, etc, $\psi(w)$ is also analytic on this Riemann surface. For large l, it is therefore possible to evaluate the integral (2.43) by steepest descents.

Let us assume that the density $\rho(d_1, d_2)$ is analytic on the contours and can be analytically continued to the saddle points on each sheet corresponding to $d_1 = -x^{-1/2}w$ and $d_2 = -x^{-1/2}w$. If the contours in (2.43) are then deformed to equivalent contours passing through these saddle points, we find that for large l

$$\langle\sigma_0\sigma_l\rangle - \langle\sigma_0\rangle\langle\sigma_l\rangle \sim \exp(-l\xi^{-1})[A + B\cos(\pi l + \delta)] \tag{2.44}$$

where the correlation length ξ is given by

$$-\xi^{-1} = 2\ln|\psi(-x^{1/2})| = \ln k(x) \tag{2.45}$$

and $k(x)$ is the elliptic modulus given by (2.29).

Combining (2.45) with (2.28), we obtain the exact relation

$$\beta\sigma\xi = \tfrac{1}{2}. \tag{2.46}$$

It follows that the correlation length diverges as $x \to 1-$, i.e. at the tricritical point, with an exponent

$$\nu = \tfrac{5}{4}. \tag{2.47}$$

3. Transfer matrix eigenvalues: regime IV

3.1. The eigenvalue spectrum

In regimes I and IV the functional equations for the eigenvalues of the transfer matrix $V(w)$ take the form

$$T(w)T(x^3w) = 1 + T(x^4w), \qquad T(x^5w) = T(w), \tag{3.1a, b}$$

where

$$T(w) = [-f(x^2)f(xw)/wf(x^3w)f(x^4w)]^N V(w) \tag{3.1c}$$

and the elliptic function $f(w) \equiv f(w, x^5)$ is given by (2.2). This time the parameters x and w are related to the interactions L and M by the equations

$$e^L = -\frac{wf(x^2)f(xw)f(x^4w)}{xf(x)f^2(w)}, \qquad e^M = \frac{f(x^2)f(xw)f(x^3w)}{f(x)f^2(x^2w)}. \tag{3.2a, b}$$

In the physical regimes x and w lie in the ranges

$$\text{Regime I} \qquad -1 < x < 0, \qquad x^2 < w < 1, \tag{3.3a}$$

$$\text{Regime IV} \qquad -1 < x < 0, \qquad 1 < w < x^{-2}. \tag{3.3b}$$

The solutions to the eigenvalue equations (3.1) are the same in regimes I and IV. Perturbation expansions, however, suggest that for the *largest* eigenvalues in regime IV

$$V(w) = O(w^{N/2}) \qquad \text{as } N \to \infty, \qquad 1 < |w| < |x|^{-2}. \tag{3.4}$$

Using this in (3.1) we find, as in regime I, that for large N

$$\begin{aligned} |T(x^4w)| &= O(|x|^{-\varepsilon N}) \gg 1, \qquad x^4 < |w| < |x|, \\ |T(x^4w)| &= O(|x|^{\varepsilon N}) \ll 1, \qquad |x| < |w| < |x|^{-1}, \end{aligned} \tag{3.5}$$

with $\varepsilon > 0$ so that one of the terms on the RHS of (3.1a) is exponentially larger than the other. From the regime I results (Baxter and Pearce 1982), it now follows that the largest eigenvalues in regime IV are of the form

$$V_s(w) = Rw^{(N-s)/2} \prod_{j=1}^{(N-s)/2} f(x^3w/a_j)f(x^4w/a_j) \prod_{k=1}^{s} f(xw/b_k) \tag{3.6}$$

where R is complex, s is a non-negative integer satisfying

$$s = N \pmod 2, \qquad s = O(1) \qquad \text{as } N \to \infty, \tag{3.7}$$

and the complex numbers a_j and b_k, locating the zeros of $V_s(w)$, are unimodular in the limit $N \to \infty$. The index s labels bands of eigenvalues.

To solve the transfer matrix equations (3.1) in the limit $N \to \infty$, we define $L(w)$ in the annulus $|x|<|w|<|x|^{-3}$ by

$$T_s(w) = w^{-(N+s)/2} L(w) \prod_{k=1}^{s} (1-xw/b_k)(1-xw)^N \tag{3.8}$$

so that $\ln L(w)$ is analytic therein. Now consider the smaller annulus $|x|<|w|<|x|^{-1}$. In this annulus (3.1a) then becomes

$$T(w)T(w/x^2) = 1, \qquad |x|<|w|<|x|^{-1}, \tag{3.9}$$

if we ignore the exponentially small term on the RHS. Taking logarithms, Laurent expanding and solving in the usual way, we find that

$$T_s(w) = \phi^N(w) \prod_{k=1}^{s} \phi(w/b_k), \qquad |x|<|w|<|x|^{-3}, \tag{3.10}$$

where

$$\phi(w) = w^{-1/2} f(xw, x^4)/f(xw^{-1}, x^4). \tag{3.11}$$

The solution in the remainder of the period annulus is given by the relation

$$T_s(w) = T_s(w/x)T_s(w/x^2), \qquad x^2<|w|<|x|. \tag{3.12}$$

If we set $w = b_j x^{-4}$ in the transfer matrix equation (3.1a) the LHS vanishes and

$$1 + T_s(b_j) = 0. \tag{3.13}$$

That is, since $\phi^{-1}(w) = \phi(w^{-1})$, the b_j are solutions of the equations

$$\phi^N(b_j) = -\prod_{k=1}^{s} \phi(b_k/b_j), \qquad j = 1,2,\ldots,s. \tag{3.14}$$

Since $|\phi(w)| = 1$ when $|w| = 1$, the equations are consistent with the requirement that the b_j be unimodular.

For N even, the transfer matrix $V(w)$ has two largest eigenvalues in the physical regime $1<w<x^{-2}$ given by (3.10) with $s = 0$, i.e.

$$T_{0;\pm}(w) = \pm\phi^N(w). \tag{3.15}$$

These correspond to the two possible square ordered phases and agree with the partition function per site previously calculated (Baxter 1980) for regime IV by the matrix inversion trick. For each positive integer s, such that $s = N \pmod 2$, (3.10) and (3.14) give a band of $[2N/(N+s)]\binom{(N+s)/2}{(N-s)/2}$ complex eigenvalues. This is the number of ways of placing $r = \frac{1}{2}(N-s)$ particles on a row of N sites, with periodic boundary conditions, such that no two particles are adjacent. For these bands of eigenvalues we shall assume that the solutions of (3.14) form continuous distributions on the circles $|b_1| = 1$, $|b_2| = 1$, etc with densities $\rho(b_1)$, $\rho(b_1, b_2)$, etc.

3.2. Interfacial tension

As for regime III, the interfacial tension σ in regime IV can be calculated from the asymptotic degeneracy of the pair of largest eigenvalues $T_{0;\pm}(w)$. Here, however, we shall calculate the interfacial tension by an alternative method. Let us consider a large $P \times N$ lattice with N much greater than P under the constraint $P = 0 \pmod 2$,

$N = 1 \pmod 2$. These constraints force a mismatched vertical seam into the domain structure of the model. The interfacial tension σ is then related to the excess free energy above the bulk free energy for such a system by

$$-\beta\sigma = \lim_{P\to\infty} \lim_{N\to\infty} P^{-1} \ln \mathrm{Tr}[\boldsymbol{T}(w)/T_{0;+}(w)]^P. \tag{3.16}$$

If N is odd, (3.10) and (3.14) give a band of N complex largest eigenvalues in the physical regime $1 < w < x^{-2}$ with $s = 1$. The lower bands of eigenvalues with $s > 1$ should not contribute to the RHS of (3.16) in the limit $P \to \infty$. For large P we then obtain

$$\lim_{N\to\infty} \mathrm{Tr}[\boldsymbol{T}(w)/T_{0;+}(w)]^P \sim \frac{1}{2\pi \mathrm{i}} \oint_{|b_1|=1} \frac{\mathrm{d}b_1}{b_1} \rho(b_1)\phi^P(w/b_1). \tag{3.17}$$

As before, the integral can be evaluated by steepest descents. Using the symmetry $\phi(x^2 w) = \phi(w^{-1})$, we find that the relevant saddle point of $|\phi(w)|$ occurs at $w = -x^{-1}$. Provided $\rho(b_1)$ is analytic in the annulus $|x| < |w| < |x|^{-1}$, the contour $|b_1| = 1$ in (3.17) can now be deformed to an equivalent contour passing through the saddle point $b_1 = -xw$. Doing this and using (3.16) we obtain

$$-\beta\sigma = \ln \phi(-x^{-1}) = \ln[|x|^{1/2} f(-1, x^4)/f(-x^2, x^4)] = \tfrac{1}{2} \ln k(x^2) \tag{3.18}$$

where $k(x)$ is the elliptic modulus given by (2.29).

In the limit $x \to -1+$, i.e. at the tricritical point, the interfacial tension vanishes. Transforming to the conjugate modulus (Baxter 1982, ch 15) using the equations

$$x = -\exp(-\pi^2/5\varepsilon), \qquad t = -\mathrm{e}^{-\varepsilon}, \tag{3.19}$$

(3.18) becomes

$$-\beta\sigma = \tfrac{1}{2} \ln k'(|t|^{5/2}) \tag{3.20}$$

where $k'(x)$ is the complementary elliptic modulus given by (2.32) and t is given by (1.2). It follows that near the tricritical point

$$\beta\sigma \sim 4|t|^{5/2}, \qquad t \to 0-, \tag{3.21}$$

and hence the interfacial tension exponent is

$$\mu' = \tfrac{5}{2}. \tag{3.22}$$

3.3. Correlation length

We now calculate the correlation length in the physical regime $1 < w < x^{-2}$. If N is even, $s = 0$ gives a pair $T_{0;\pm}(w)$ of largest eigenvalues. The next-largest eigenvalues are then given by setting $s = 2$. In this case (3.10) and (3.14) give a band of $N^2/4$ complex eigenvalues. Noting that $|\phi(w)| < 1$ for $1 < |w| < x^{-2}$, we see that the gap between the largest and next-largest eigenvalues is

$$T_1(w)/T_{0;+}(w) = \phi(w/b_1)\phi(w/b_2). \tag{3.23}$$

If we now repeat the arguments outlined for calculating the correlations in the other regimes, we find that the asymptotic behaviour of the correlation between sites separated by l rows, for l large, is

$$\langle\sigma_0\sigma_l\rangle - \langle\sigma_0\rangle\langle\sigma_l\rangle \sim \left(\frac{1}{4\pi \mathrm{i}}\right)^2 \oint_{|b_1|=1} \frac{\mathrm{d}b_1}{b_1} \oint_{|b_2|=1} \frac{\mathrm{d}b_2}{b_2} \rho(b_1, b_2)\phi^l(w/b_1)\phi^l(w/b_2). \tag{3.24}$$

Here again the contours are closed loops formed by two circuits about the origin on the Riemann surface of two sheets on which the square root function is analytic. Under the usual assumptions, the contours in (3.24) can be deformed to equivalent contours passing through the saddle points on each sheet corresponding to $b_1 = -xw$, $b_2 = -xw$. Evaluating the integral by steepest descents, we then find that for large l

$$\langle\sigma_0\sigma_l\rangle - \langle\sigma_0\rangle\langle\sigma_l\rangle \sim \exp(-l\xi^{-1})[A + B\cos(\pi l + \delta)] \tag{3.25}$$

where the correlation length ξ is given by

$$-\xi^{-1} = 2\ln\phi(-x^{-1}) = \ln k(x^2). \tag{3.26}$$

Combining (3.26) with (3.18) we again obtain the relation

$$\beta\sigma\xi = \tfrac{1}{2}. \tag{3.27}$$

The correlation length therefore diverges as $x \to -1+$, i.e. at the tricritical point, with an exponent

$$\nu' = \tfrac{5}{2}. \tag{3.28}$$

4. Sublattice densities

The sublattice densities ρ_k of the generalised hard hexagon model can be calculated using corner transfer matrices (Baxter 1981a). An intriguing feature of the calculations is that various Rogers–Ramanujan-type identities occur naturally in the working (Baxter 1981b), involving the functions

$$G(x) = \prod_{n=1}^{\infty}[(1-x^{5n-4})(1-x^{5n-1})]^{-1}, \qquad H(x) = \prod_{n=1}^{\infty}[(1-x^{5n-3})(1-x^{5n-2})]^{-1},$$
$$Q(x) = \prod_{n=1}^{\infty}(1-x^n), \qquad P(x) = \prod_{n=1}^{\infty}(1-x^{2n-1}). \tag{4.1}$$

4.1. Regime III

If regime III is approached from the fluid phase, then the system is homogeneous. This is the case that has been considered previously (Baxter 1980, 1981b, 1982). There is just one density ρ_f, given by

$$\rho_f = \sum_{\tau}\sigma_1 r_0^{2\sigma_1} q^{\phi(\tau)} \Big/ \sum_{\tau} r_0^{2\sigma_1} q^{\phi(\tau)}, \tag{4.2}$$

where $q = x^2$,

$$r_0^2 = xH(x)/G(x), \tag{4.3}$$

$$\phi(\tau) = \sum_{j=1}^{m} j(\sigma_{j+1} - \sigma_j\sigma_{j+2} - s_{j+1} + s_j s_{j+2}), \tag{4.4}$$

and τ is the set of m 'spins' (or occupation numbers) $\sigma_1, \sigma_2, \ldots, \sigma_m$. Each σ_j can take the values 0 and 1, but two adjacent spins cannot both be 1, i.e.

$$\sigma_j\sigma_{j+1} = 0 \qquad \text{for } j = 1,2,\ldots,m+1. \tag{4.5}$$

The s_j are the 'ground state' values of the σ_j, and for the fluid phase they have the values

$$s_j = 0 \qquad \text{for } j \geqslant 1. \tag{4.6}$$

The expression (4.4) involves the 'boundary' spins σ_{m+1}, σ_{m+2}, which are to be fixed at their ground state values:

$$\sigma_{m+1} = s_{m+1}, \qquad \sigma_{m+2} = s_{m+2}. \tag{4.7}$$

The summations in (4.2) are over all values of $\tau = \{\sigma_1, \ldots, \sigma_m\}$, subject to the restriction (4.5). Ultimately m is to become infinite.

It is convenient to define

$$X_m^{abc} = \sum_{\sigma_2, \ldots, \sigma_m} q^{\sum j(\sigma_{j+1} - \sigma_j \sigma_{j+2})}, \tag{4.8a}$$

where the summation in the exponent is over $j = 1, \ldots, m$, and σ_1, σ_{m+1}, σ_{m+2} are given the values

$$\sigma_1 = a, \qquad \sigma_{m+1} = b, \qquad \sigma_{m+2} = c. \tag{4.8b}$$

Then by performing the σ_1 summations explicitly, (4.2) can be written as

$$\rho_f = xH(x)F(1)/[G(x)F(0) + xH(x)F(1)], \tag{4.9}$$

where

$$F(a) = X_m^{a00}. \tag{4.10}$$

From (4.8) it is easily seen that X_m^{abc} satisfies the recurrence relations

$$X_m^{a00} = X_{m-1}^{a00} + X_{m-1}^{a10}, \qquad X_m^{a10} = q^m X_{m-1}^{a01}, \qquad X_m^{a01} = X_{m-1}^{a00} + q^{-m} X_{m-1}^{a10}. \tag{4.11}$$

In the previous calculations by one of us (Baxter 1981b, 1982), m was taken to be infinite as soon as possible. However, Andrews (1981) has shown that X_m^{a00} can be calculated conveniently even for m finite, in fact

$$\begin{aligned} X_m^{000} &= \sum_{n,r \geqslant 0} q^{n(3n+1)/2} q^r \begin{bmatrix} m-2n-2r \\ n \end{bmatrix}_q \begin{bmatrix} r+n \\ r \end{bmatrix}_{q^2} \\ &= \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{5\lambda^2+\lambda} \begin{bmatrix} m \\ a_{m-5\lambda} \end{bmatrix}_q, \end{aligned} \tag{4.12a}$$

$$\begin{aligned} X_m^{100} &= \sum_{n,r \geqslant 0} q^{3n(n+1)/2} q^r \begin{bmatrix} m-2n-2r-1 \\ n \end{bmatrix}_q \begin{bmatrix} r+n \\ r \end{bmatrix}_{q^2} \\ &= \sum_{\lambda=-\infty}^{\infty} (-1)^\lambda q^{5\lambda^2-3\lambda} \begin{bmatrix} m \\ 1+a_{m-5\lambda} \end{bmatrix}_q, \end{aligned} \tag{4.12b}$$

where a_m is the largest integer not exceeding $\frac{1}{2}m$, and

$$\begin{aligned} \begin{bmatrix} N \\ M \end{bmatrix}_q &= \frac{(1-q^N)(1-q^{N-1}) \ldots (1-q^{N-M+1})}{(1-q^M)(1-q^{M-1}) \ldots (1-q)} \qquad \text{if } M \geqslant 0, \\ &= 0 \qquad \text{if } M < 0 \text{ and } N \geqslant 0. \end{aligned} \tag{4.13}$$

Now taking the limit $m \to \infty$, noting that $|q| < 1$, the r-summations can be performed in (4.12), while the λ-summations can be performed by using the identity

$$f(w, q) = \sum_{\lambda=-\infty}^{\infty} (-1)^{\lambda} q^{\lambda(\lambda+1)/2} w^{-\lambda}, \tag{4.14}$$

$f(w, q)$ being defined by (2.2). Then (4.12) and (4.10) give

$$F(0) = \sum_{n=0}^{\infty} q^{n(3n+1)/2} / [(q)_n (q; q^2)_{n+1}] = G(q^2) Q(q^2)/Q(q), \tag{4.15a}$$

$$F(1) = \sum_{n=0}^{\infty} q^{3n(n+1)/2} / [(q)_n (q; q^2)_{n+1}] = H(q^2) Q(q^2)/Q(q), \tag{4.15b}$$

where

$$(a; q)_n = \prod_{j=0}^{n-1} (1 - aq^j) \tag{4.16}$$

$$(q)_n = (q; q)_n = \prod_{j=1}^{n} (1 - q^j). \tag{4.17}$$

Thus $F(0)$ and $F(1)$ can each be written either as an infinite sum or as a product. The equivalence of the two forms is a mathematical identity of Rogers–Ramanujan type (equations (46) and (44) of Slater (1951)). Substituting the product forms into (4.9) and using Rogers' identity

$$G(x)G(x^4) + xH(x)H(x^4) = [P(-x)]^2, \tag{4.18}$$

(equation (2) of Birch (1975)), we find that

$$\rho_f = xH(x)H(x^4)/[P(-x)]^2, \tag{4.19}$$

which is the result previously reported (Baxter 1981b).

As Huse has pointed out, regime III can also be approached from the solid phase. The system is then inhomogeneous, having one density (ρ_1) on one sublattice, another density (ρ_2) on the other. These sublattice densities are again given by (4.2)–(4.7), with ρ_f replaced by ρ_k, the only difference being that now we must use the ground state appropriate to the ordered state of the square lattice, i.e. (4.6) is to be replaced by

$$s_{2j+k} = 1, \qquad s_{2j+k+1} = 0, \tag{4.20}$$

for all integers j. Here k determines which ordered state is being considered, and which sublattice density is being calculated: it is either 1 or 2. (These ground states (4.20) are the same as those of regime IV.)

Using these ground-state spin values in (4.4) and (4.7), we obtain equations analogous to (4.9):

$$\rho_k = xH(x)F_k(1)/[G(x)F_k(0) + xH(x)F_k(1)] \tag{4.21}$$

for $k = 1,2$, where $F_k(0)$ and $F_k(1)$ are given by

$$F_1(a) = q^{-m/2} X_m^{a10}, \qquad F_2(a) = q^{m/2} X_m^{a01} \qquad \text{if } m \text{ even}, \tag{4.22a}$$

$$F_1(a) = q^{(m+1)/2} X_m^{a01}, \qquad F_2(a) = q^{-(m+1)/2} X_m^{a10} \qquad \text{if } m \text{ odd}. \tag{4.22b}$$

Comparing (4.9) and (4.10) with (4.21) and (4.22), we see that the three densities ρ_f, ρ_1, ρ_2 correspond to the three functions X_m^{a00}, X_m^{a10} and X_m^{a01}. These in turn correspond to the three choices of the boundary condition (4.7), so Huse's observation neatly and naturally completes the set of equations.

We can calculate X_m^{a10} and X_m^{a01} from the first two equations in (4.11), using Andrews' results (4.12). Substituting the results into (4.22), then letting $m \to \infty$, we obtain

$$F_1(0) = \sum_{n=1}^{\infty} \sum_{k \geqslant (3n-1)/2} \frac{q^{n(3n-5)/2+k+1}}{(q^2)_n} \begin{bmatrix} 2k-2n \\ n-1 \end{bmatrix}_q$$
$$= [qf(-q^{13}, q^{40}) - q^4 f(-q^3, q^{40})]/Q(q), \tag{4.23a}$$

$$F_1(1) = 1 + q \sum_{n=1}^{\infty} \sum_{k \geqslant (3n-2)/2} \frac{q^{n(3n-3)/2+k}}{(q^2)_n} \begin{bmatrix} 2k+1-2n \\ n-1 \end{bmatrix}_q$$
$$= [f(-q^{19}, q^{40}) - qf(-q^{11}, q^{40})]/Q(q), \tag{4.23b}$$

$$F_2(0) = 1 + \sum_{n=1}^{\infty} \sum_{k \geqslant 3n/2} \frac{q^{n(3n-5)/2+k}}{(q^2)_n} \begin{bmatrix} 2k-2n-1 \\ n-1 \end{bmatrix}_q$$
$$= [f(-q^{17}, q^{40}) - q^2 f(-q^7, q^{40})]/Q(q), \tag{4.23c}$$

$$F_2(1) = \sum_{n=1}^{\infty} \sum_{k \geqslant (3n-1)/2} \frac{q^{n(3n-3)/2+k}}{(q^2)_n} \begin{bmatrix} 2k-2n \\ n-1 \end{bmatrix}_q$$
$$= [qf(-q^9, q^{40}) - q^4 f(-q, q^{40})]/Q(q). \tag{4.23d}$$

In each case the second form, involving two functions f, comes from the λ-series in (4.12). Each is a series divided by $Q(q)$; equation (4.14) has been used to write the series in closed form. (Even values of λ in (4.12) give the first f-function, odd values the second.)

These series can be rearranged. Noting that $Q(x)G(x) = f(x^2, x^5)$, $Q(x)H(x) = f(x, x^5)$ and using $q = x^2$ and (4.14), we find that

$$\begin{aligned} F_1(0) &= \tfrac{1}{2}x[Q(-x)H(-x) - Q(x)H(x)]/Q(x^2), \\ F_1(1) &= \tfrac{1}{2}[Q(x)G(x) + Q(-x)G(-x)]/Q(x^2), \\ F_2(0) &= \tfrac{1}{2}[Q(x)H(x) + Q(-x)H(-x)]/Q(x^2), \\ F_2(1) &= \tfrac{1}{2}x^{-1}[Q(-x)G(-x) - Q(x)G(x)]/Q(x^2). \end{aligned} \tag{4.24}$$

Thus again we can write the results very neatly in terms of the functions G, H and Q. Substituting them into (4.21) and using the identity

$$G(x)H(-x) + G(-x)H(x) = 2/[P(x^2)]^2 \tag{4.25}$$

(equation (23) of Birch (1975): this identity was stated by Ramanujan and proved by Watson (1933)), we obtain

$$\begin{aligned} \rho_1 &= \tfrac{1}{2}H(x)[P(x^2)]^2[Q(x)G(x) + Q(-x)G(-x)]/Q(-x), \\ \rho_2 &= \tfrac{1}{2}H(x)[P(x^2)]^2[Q(-x)G(-x) - Q(x)G(x)]/Q(-x). \end{aligned} \tag{4.26}$$

Thus the mean total density in the solid phase of regime III is

$$\rho_m = \tfrac{1}{2}(\rho_1 + \rho_2) = \tfrac{1}{2}G(-x)H(x)[P(x^2)]^2, \tag{4.27}$$

and the order parameter is

$$R = \rho_1 - \rho_2 = G(x)H(x)Q(x)[P(x^2)]^2/Q(-x) = Q(x)Q(x^5)/[Q(x^2)Q(x^4)]. \tag{4.28}$$

To obtain the behaviour near the tricritical point, i.e. as $x \to 1-$, we again make a transformation to the conjugate nome t defined by (1.2) and (2.30). Defining

$$H_1(x) = (2\sin\tfrac{2}{5}\pi)^{-1} \prod_{n=1}^{\infty} (1 - 2x^n \cos\tfrac{4}{5}\pi + x^{2n})^{-1}, \tag{4.29}$$

and using standard elliptic function identities (Baxter 1982, pp 445–6), we obtain

$$\rho_f = H_1(t)H_1(t^{1/4})/[P(-t^{5/4})]^2 = \rho_c - 5^{-1/2}t^{1/4} - \tfrac{1}{2}(\sqrt{5}-1)t + O(t^{5/4}), \tag{4.30}$$

$$\rho_m = H_1(t)H_1(-t^{1/4})/[P(t^{5/4})]^2 = \rho_c + 5^{-1/2}t^{1/4} - \tfrac{1}{2}(\sqrt{5}-1)t + O(t^{5/4}), \tag{4.31}$$

$$R = (8/5)^{1/2}t^{3/32}Q(t)Q(t^5)/[Q(t^{5/4})Q(t^{5/2})] = (8/5)^{1/2}t^{3/32}[1 - t + t^{5/4} + O(t^2)], \tag{4.32}$$

where

$$\rho_c = (5-\sqrt{5})/10 = 0.27639\ldots \tag{4.33}$$

is the density at the tricritical point. Since t is the 'deviation from criticality' variable, we see at once that we can define two tricritical exponents β_1, β_2 by

$$R \sim t^{\beta_1}, \qquad \Delta\rho = \rho_m - \rho_f \sim t^{\beta_2}, \tag{4.34}$$

and that

$$\beta_1 = \tfrac{3}{32}, \qquad \beta_2 = \tfrac{1}{4}. \tag{4.35}$$

Here β_1 corresponds to the difference between the sublattice densities in the solid phase; β_2 corresponds to the density discontinuity between the fluid and solid phases.

It is interesting to note from (4.30) and (4.31) that ρ_f and ρ_m differ only in the sign of $t^{1/4}$, so are analytic continuations of one another. This is despite the fact that their derivations, and the x-expressions (4.19) and (4.27), seem to have no particular connection.

4.2. Regime IV

The two sublattice densities in regime IV have been obtained previously (Baxter 1981b, 1982). For completeness we list the results:

$$\begin{aligned}\rho_m &= \tfrac{1}{2}(\rho_1 + \rho_2) = \tfrac{1}{2}G(x)H(-x)[P(x^2)]^2 = H_1(t)H_1(t^4)/[P(-t^5)]^2 \\ &= \rho_c - 5^{-1/2}t - \tfrac{1}{2}(\sqrt{5}-1)t^4 + O(t^5)\end{aligned} \tag{4.36}$$

$$\begin{aligned}R &= \rho_1 - \rho_2 = Q^2(x^2)Q(x^5)/[Q(x)Q^2(x^4)] \\ &= (4/5)^{1/2}(-t)^{1/4}Q(t)Q^2(t^{10})/[Q(t^5)Q^2(-t^5)] \\ &= (4/5)^{1/2}(-t)^{1/4}[1 - t - t^2 + O(t^4)].\end{aligned} \tag{4.37}$$

(The first terms in these expansions are given in equation (47) of Baxter (1980), and equation (105) of Baxter (1981b), q^2 and p therein being equal to t. There is an error in those equations where they give ρ for regime IV: $5^{1/2}$ should be replaced by $5^{-1/2}$, which then makes them consistent with (4.36).)

These parameters x and t are the same as those used in § 3 and equation (1.2). They are negative, between -1 and 0, and are related by (3.19). From (1.2), t is the 'deviation from criticality' variable, so we see that in regime IV the exponent β associated with the sublattice density difference has the value $\frac{1}{4}$, while in regime III it is $\frac{3}{32}$. Presumably this difference is connected with Huse's observation that regime III is the boundary between the fluid and solid phases of the interacting hard squares system, while regime IV lies inside the solid phase.

There are some unexpected similarities between the results for regimes III and IV. Examining the expressions (4.27) and (4.36) for ρ_m in terms of the appropriate variable x, we see that they differ only in that x is replaced by $-x$. A corollary of this is that the t-expressions (4.31) and (4.36) differ only in that $-t^{1/4}$ is replaced by t. The identity (4.25) also occurs in the working for regime IV (equation (94) of Baxter (1981b), and equation (14.5.53) of Baxter (1982)).

References

Andrews G E 1981 *Proc. Natl Acad. Sci. USA* **78** 5290–2
Baxter R J 1980 *J. Phys. A: Math. Gen.* **13** L61–70
——1981a *Physica* **106A** 18–27
——1981b *J. Stat. Phys.* **26** 427–52
——1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
Baxter R J, Enting I G and Tsang S K 1980 *J. Stat. Phys.* **22** 465–89
Baxter R J and Pearce P A 1982 *J. Phys. A: Math. Gen.* **15** 897–910
Birch B J 1975 *Math. Proc. Camb. Phil. Soc.* **78** 73–9
Fisher M E 1969 *J. Phys. Soc. Japan, Suppl.* **26** 87–8
Gaunt D S and Fisher M E 1965 *J. Chem. Phys.* **43** 2840–63
Huse D A 1982 *Phys. Rev. Lett.* **49** 1121–4
Slater L J 1951 *Proc. London Math. Soc.* (2) **54** 147–67
Watson G N 1933 *J. Indian Math. Soc.* **20** 57–69